String Theory and Integrable Lattice Models

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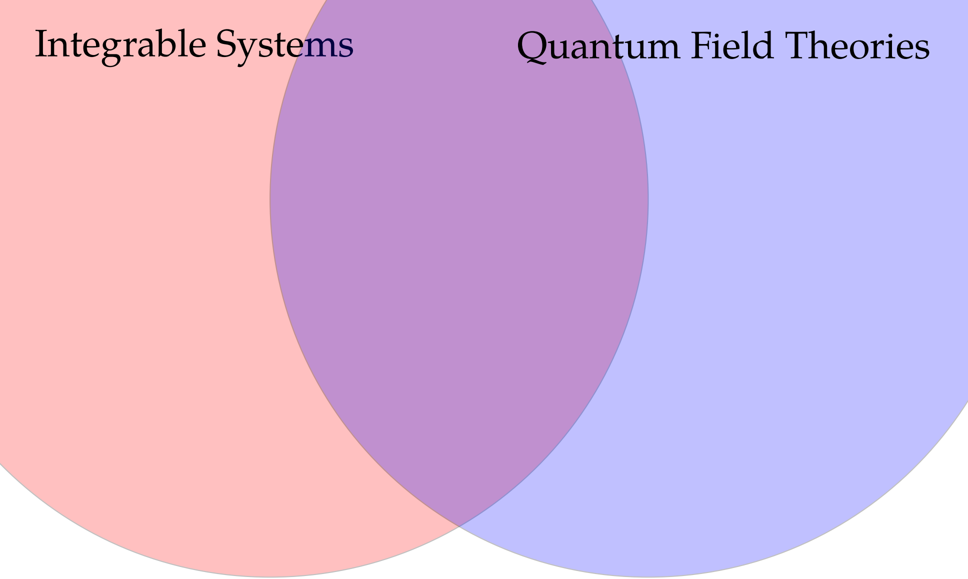
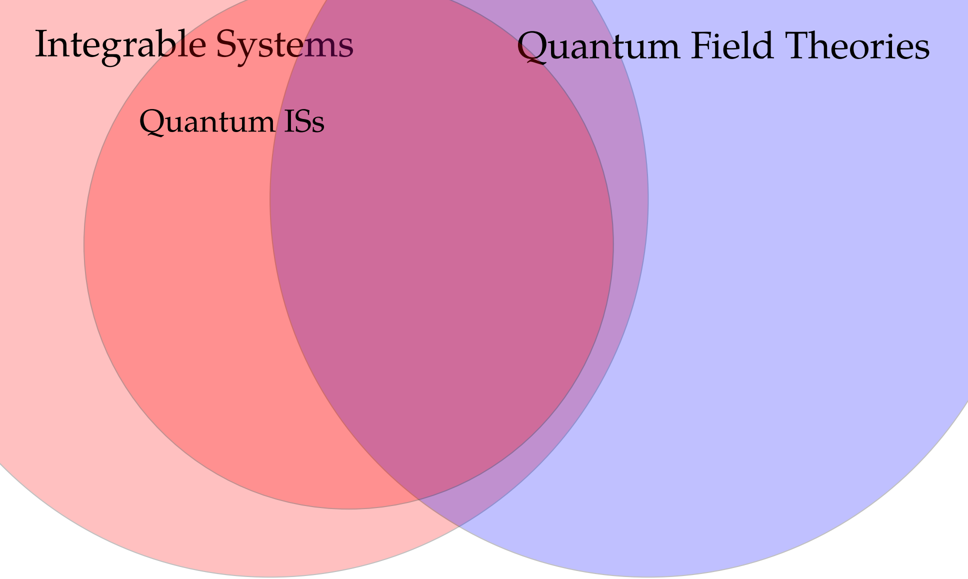
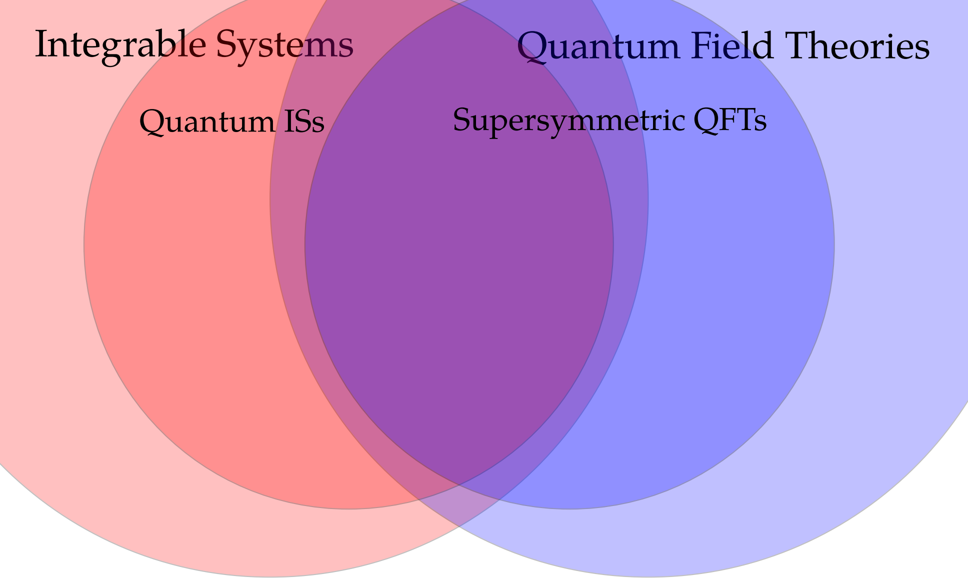
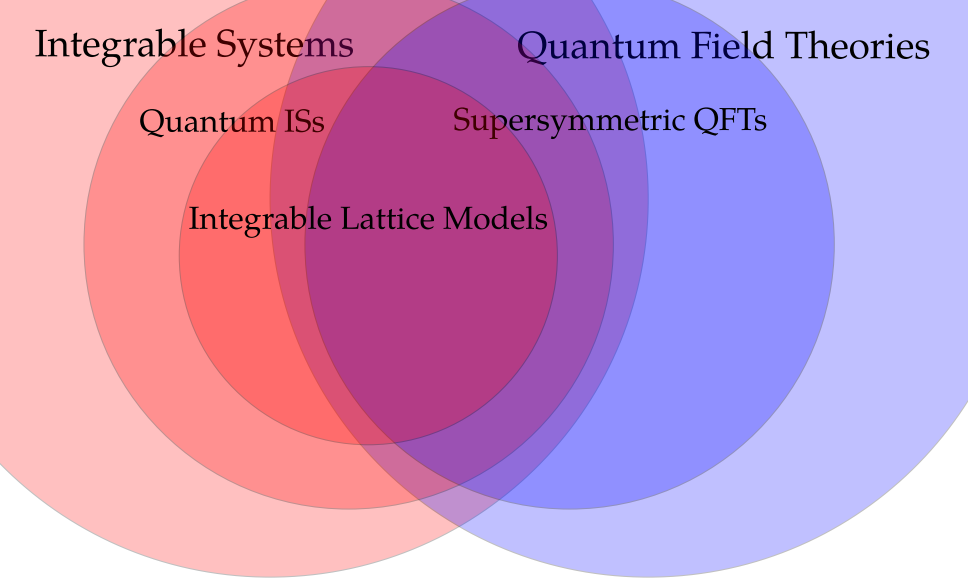
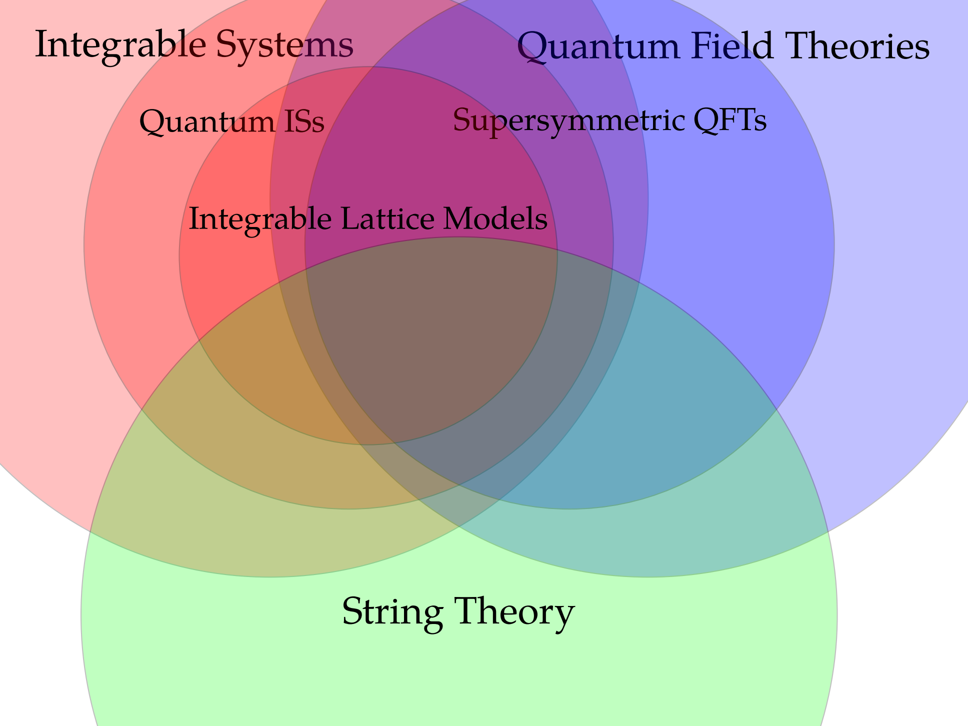
Integrable Systems

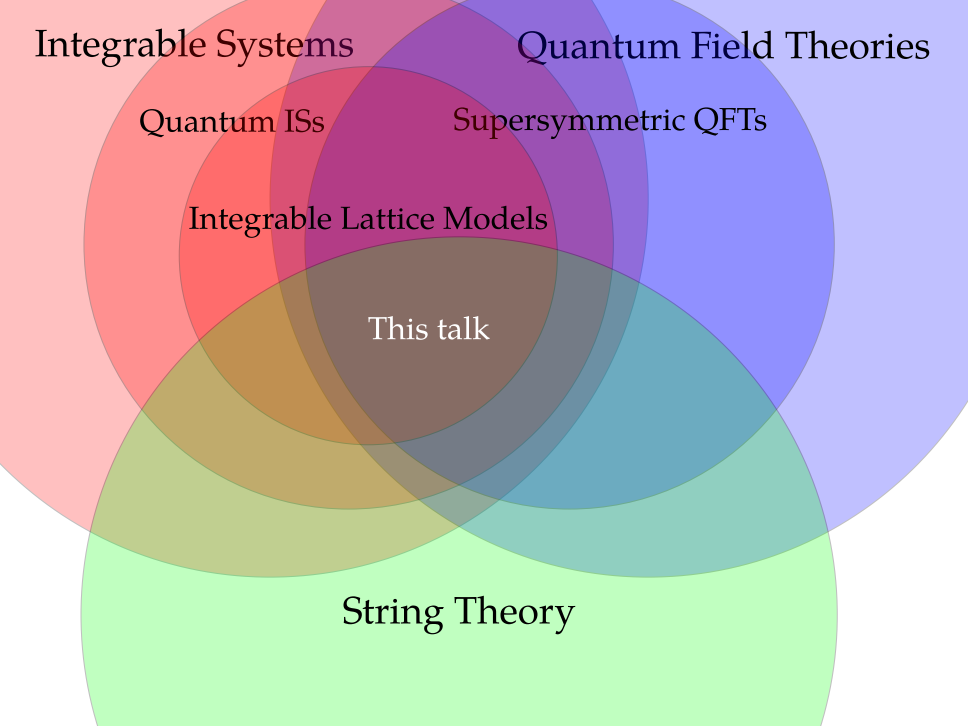

Integrable Systems
Quantum Field Theories


Integrable Systems
Quantum Field Theories
Quantum ISs


Integrable Systems
Quantum Field Theories
Quantum ISs
Supersymmetric QFTs


Integrable Systems
Quantum Field Theories
Quantum ISs
Supersymmetric QFTs
Integrable Lattice Models


Integrable Systems
Quantum Field Theories
Quantum ISs
Supersymmetric QFTs
Integrable Lattice Models
String Theory


Integrable Systems
Quantum Field Theories
Quantum ISs
Supersymmetric QFTs
Integrable Lattice Models
This talk
String Theory

Integrable lattice models can be embedded in String Theory.

What does this buy us?

A lot!

Today:

- Construct an integrable lattice model that unifies
    - Belavin model
    - Jimbo–Miwa–Okado model group
    - Bazhanov–Sergeev model
- Relate it to 4d SUSY QFTs.

Fix

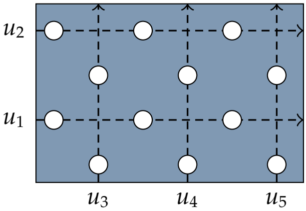
- $N \in \mathbb{N},\ N \geq 2$
- $\tau, \gamma \in \mathbb{C},\ \operatorname{Im} \tau, \operatorname{Im} \gamma > 0$

Let

- $V = \mathbb{C}^N$
- $\mathbb{V} = \{\text{meromorphic symmetric functions of } (z_1, \ldots, z_N)\}$

Belavin model [Belavin '81]

Belavin model is a lattice model in Statistical Mechanics:



It's a vertex model: spins ○ live on edges, interact at vertices.

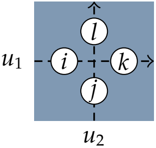
Spin variables $i, j, k, l, \ldots \in \{1, \ldots, N\}$.

Each line carries a spectral parameter $u \in \mathbb{C}$.

Lattice can be drawn on any surface Σ; in this talk $\Sigma = T^2$.

Interaction is governed by the R-matrix $R^{\mathrm{B}}(u) \in \mathrm{End}(V \otimes V)$.

Local Boltzmann weight:



$$= R^{\mathrm{B}}(u_{12})_{ij}^{kl}\,, \qquad u_{12} = u_1 - u_2\,.$$

Partition function

$$Z = \sum_{\text{spin configs}} \prod_{\text{vertices}} \text{local Boltzmann weight}\,.$$

Explicitly,

$$R^{\mathrm{B}}(u)_{ij}^{kl} = \delta_{i+j,k+l} \frac{\theta_1(\gamma)}{\theta_1(u+\gamma)} \frac{\theta^{(k-l)}(u+\gamma)}{\theta^{(k-i)}(\gamma)\theta^{(i-l)}(u)} \frac{\prod_{m=0}^{N-1} \theta^{(m)}(u)}{\prod_{n=1}^{N-1} \theta^{(n)}(0)} ,$$

where i, j, k, l are treated mod N and

$$\theta\left[\begin{array}{c} a \\ b \end{array}\right](u|\tau) = \sum_{n=-\infty}^{\infty} e^{\pi \mathrm{i}(n+a)^2\tau + 2\pi\mathrm{i}(n+a)(u+b)} ,$$

$$\theta^{(j)}(u) = \theta\left[\begin{array}{c} 1/2 - j/N \\ 1/2 \end{array}\right](u|N\tau) ,$$

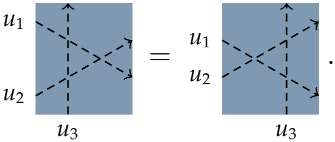
$$\theta_1(u) = -\theta\left[\begin{array}{c} 1/2 \\ 1/2 \end{array}\right](u|\tau) .$$

When $N = 2$, we get the 8-vertex model.

R^{B} satisfies the Yang–Baxter equation

$$R^{\mathrm{B}}_{12}(u_{12})R^{\mathrm{B}}_{13}(u_{13})R^{\mathrm{B}}_{23}(u_{23}) = R^{\mathrm{B}}_{23}(u_{23})R^{\mathrm{B}}_{13}(u_{13})R^{\mathrm{B}}_{12}(u_{12})\,.$$

Graphically,



It follows the model is integrable.

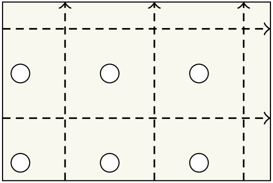
This is quantum integrability:

$$\text{2d lattice model} \leftrightarrow \text{1d quantum spin chain}\,.$$

Belavin model $\leftrightarrow$ $\mathfrak{sl}_N$ generalization of XYZ spin chain.

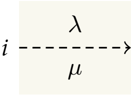
Jimbo–Miwa–Okado model [JMO '87]

JMO model is an IRF model, i.e. spins live on faces:



Spin variables $\lambda \in \mathfrak{h} \subset \mathfrak{sl}_N$: $\lambda = (\lambda_1, \ldots, \lambda_N)$, $\sum_{i=1}^N \lambda_i = 0$.

Allowed configs: to each edge we can assign $i \in \{1, \ldots, N\}$ s.t.



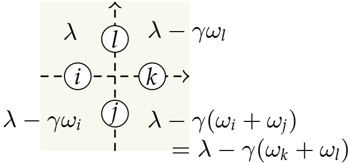
$$, \quad \mu = \lambda - \gamma w_i \,.$$

$w_i = e_i - \frac{1}{N}\sum_{j=1}^N e_j$ is the weight of $e_i \in V = \mathbb{C}^N$, $(e_i)_j = \delta_{ij}$.

Alternatively, think of $i, j, \ldots$ as spin variables living on edges.

We get a vertex model with a dynamical variable λ.

Specify λ on one face. Spin config determines it on the rest:



This vertex model is described by Felder's R-matrix $R^{\mathrm{F}}(u, \lambda)$:

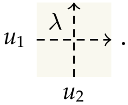
$$(R^{\mathrm{F}})^{ii}_{ii} = 1\,, \quad (R^{\mathrm{F}})^{ij}_{ij} = \frac{\theta_1(u)\theta_1(\lambda_{ij} + \gamma)}{\theta_1(u+\gamma)\theta_1(\lambda_{ij})}\,, \quad (R^{\mathrm{F}})^{ji}_{ij} = \frac{\theta_1(\gamma)\theta_1(u + \lambda_{ij})}{\theta_1(u+\gamma)\theta_1(\lambda_{ij})}$$

[Felder '94, Felder–Varchenko '97].

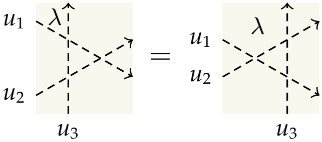
For $N = 2$, we get the 8VSOS model [Baxter '73].

Graphically represent R^{F} as

$$R^{\mathrm{F}}(u_{12}, \lambda) =$$



This time, we have the dynamical YBE:



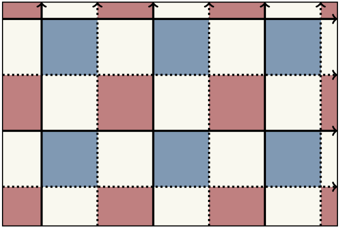
or

$$\begin{aligned} &R^{\mathrm{F}}_{12}(u_{12}, \lambda - \gamma h_3) R^{\mathrm{F}}_{13}(u_{13}, \lambda) R^{\mathrm{F}}_{23}(u_{23}, \lambda - \gamma h_1) \\ &\qquad = R^{\mathrm{F}}_{23}(u_{23}, \lambda) R^{\mathrm{F}}_{13}(u_{13}, \lambda - \gamma h_2) R^{\mathrm{F}}_{12}(u_{12}, \lambda) \,. \end{aligned}$$

h_a: the weight of the state on the ath line.

Bazhanov–Sergeev model [BS '10, '11]

BS model is defined on a tricolor checkerboard lattice:



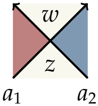
Lines carry multiplicative spectral parameters $a, b, \ldots$.

Spin variables: multiplicative dynamical variables $z, w, \ldots$ on uncolored faces. They are all independent.

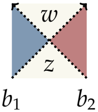
Boltzmann weights involve the elliptic gamma function

$$\Gamma(z) = \prod_{m,n=0}^{\infty} \frac{1 - p^{m+1}q^{n+1}/z}{1 - p^m q^n z}\,; \quad p = e^{2\pi i \tau}\,,\ q = e^{2\pi i \gamma}\,.$$

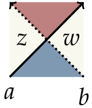
We assign Boltzmann weights



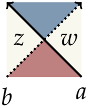
$$= M\left(\frac{a_2}{a_1}; z, w\right),$$



$$= M\left(\frac{b_2}{b_1}; w, z\right)$$



$$= D\left(\frac{b}{a}; z, w\right),$$



$$= D\left(\frac{a}{b}; w, z\right),$$

where

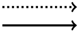
$$M(a; z, w) = \prod_{i,j} \Gamma\left(a\frac{w_i}{z_j}\right) \Big/ \Gamma\left(a^N\right), \quad D(a; z, w) = \prod_{i,j} \Gamma\left(\sqrt{pq}\frac{1}{a}\frac{w_i}{z_j}\right).$$

Each spin z is integrated over $\mathbb{T}^{N-1} \subset \mathrm{SU}(N)$ with measure

$$\frac{(p;p)_\infty^{N-1}(q;q)_\infty^{N-1}}{N!} \prod_{k=1}^{N-1} \frac{\mathrm{d}z_k}{2\pi \mathrm{i} z_k} \prod_{i \neq j} \frac{1}{\Gamma(z_i/z_j)}; \quad (p;p)_\infty = \prod_{k=1}^{\infty}(1-p^k).$$

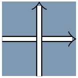
$M(a; z, w)$ defines the elliptic Fourier transform on $f(z_1, \ldots, z_N)$ [Spiridonov '03, Spiridonov–Warnaar '05].

We can reformulate the BS model as a vertex model.

Introduce double line notation $(a, b) \Longrightarrow \; = \; \begin{smallmatrix} b \\ a \end{smallmatrix}$  .

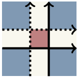
BS R-operator $R^{\mathrm{BS}}((a_1, b_1), (a_2, b_2)) \in \mathrm{End}(\mathbb{V} \otimes \mathbb{V})$ is given by

$$R^{\mathrm{BS}}\big((a_1, b_1), (a_2, b_2)\big) = (a_1, b_1)$$



(a_2, b_2)

$= \begin{smallmatrix} b_1 \\ a_1 \end{smallmatrix}$



$b_2 a_2$.

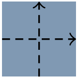
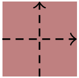
Lattice made from R^{BS} gives a tricolor checkerboard pattern.

R^{BS} is an ∞-dim R-matrix; it's an integral operator
[Derkachov–Spiridonov '12, Maruyoshi–Y '16].

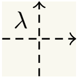
YBE follows from an integral identity for Γ [Spiridonov '03 , Rains '10].

Summary of the three models

Belavin:

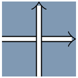
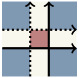
$R^{\mathrm{B}}(u_{12}) =$ u_1  u_2 $\in \mathrm{End}(V \otimes V)\,,$ $\quad \widetilde{R}^{\mathrm{B}} =$ u_1  u_2 $= (R^{\mathrm{B}})^T$

Jimbo–Miwa–Okado/Felder:

$R^{\mathrm{F}}(u_{12}, \lambda) =$ u_1 

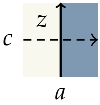
u_2 $\in \mathrm{End}(V \otimes V)$

Bazhanov–Sergeev:

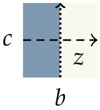
$R^{\mathrm{BS}}\big((a_1, b_1), (a_2, b_2)\big) =$ (a_1, b_1)  (a_2, b_2) $=$ $\begin{matrix} b_1 \\ a_1 \end{matrix}$  $b_2 a_2$ $\in \mathrm{End}(\mathbb{V} \otimes \mathbb{V})$

String Theory tells me the three models can be unified [Y '17].

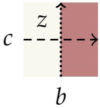
If different kinds of lines can coexist, we have more crossings:



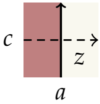
$$= S\left(\frac{c}{a}; z\right),$$



$$= S'\left(\frac{c}{b}; z\right),$$



$$= \widetilde{S}\left(\frac{c}{b}; z\right),$$

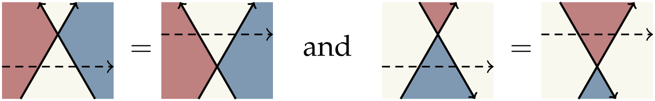


$$= \widetilde{S}'\left(\frac{c}{a}; z\right).$$

These intertwining operators are matrix-valued functions.

They must solve many Yang–Baxter equations!

Yang–Baxter equations with one dashed line such as



allow us to determine the intertwining operators:

$$S(a;z) = a^{-N/2}\Psi(u,\lambda)\,, \quad \widetilde{S}'(a;z) = a^{-N/2}Z^{N/2}\Phi(u,\lambda)^T\,,$$

$$S'(a;z) = S(\check{a};z)^{-1}\,, \qquad \widetilde{S}(a;z) = (-1)^{N-1}\widetilde{S}'(\check{a};z)^{-1}Z^N\,,$$

with $a = e^{2\pi\mathrm{i}u/N}$, $z_j = e^{2\pi\mathrm{i}\lambda_j}$, $\check{a} = q^{-1/N}\sqrt{pq}\,a$, and

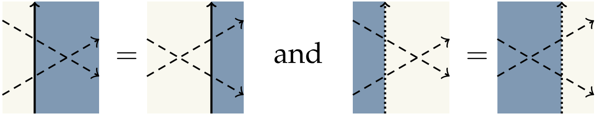
$$Z = \mathrm{diag}(z_1,\dots,z_N)\,,$$

$$\Phi(u,\lambda)_i^j = \theta^{(j)}\left(u - N\lambda_i + \frac{N-1}{2}\right),$$

$$\Psi(u,\lambda)_i^j = \Phi(u,-\lambda)_i^j \Big/ \prod_{k(\neq i)} \theta_1(\lambda_{ki})\,.$$

Similar analysis in [Sergeev '92, Quano–Fujii '93, Derkachov–Spiridonov]

Yang–Baxter equations with two dashed lines such as



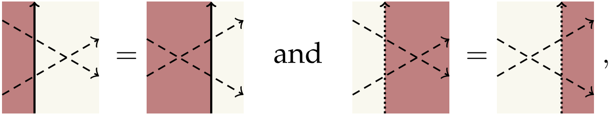
or

$$R^{\mathrm{B}} S_1 S_2 = S_2 S_1 R^{\mathrm{F}} \quad \text{and} \quad R^{\mathrm{F}} S'_1 S'_2 = S'_2 S'_1 R^{\mathrm{B}}$$

describe the vertex–face correspondence [Baxter '73, JMO].

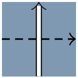
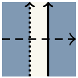
They relate R^{B} and R^{F}, vertex and IRF models.

Those in different colors,



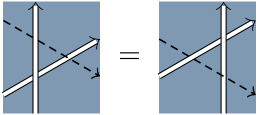
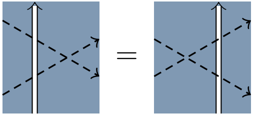
also hold.

We can also construct an L-operator

$L^{\mathrm{B}} =$  $=$  $\in \mathrm{End}(V \otimes \mathbb{V})\,.$

It's a matrix of difference operators [Hasegawa '90, Sergeev, Quano–Fujii].

It satisfies two RLL relations, one with R^{B} and another with R^{BS}:

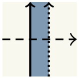
 and 

or

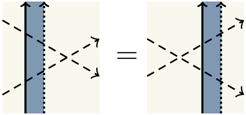
$$L_1^{\mathrm{B}} L_2^{\mathrm{B}} R^{\mathrm{BS}} = R^{\mathrm{BS}} L_2^{\mathrm{B}} L_1^{\mathrm{B}} \quad \text{and} \quad R^{\mathrm{B}} L_1^{\mathrm{B}} L_2^{\mathrm{B}} = L_2^{\mathrm{B}} L_1^{\mathrm{B}} R^{\mathrm{B}}\,.$$

This is the elliptic lift of the chiral Potts/six-vertex relation [Bazhanov–Stroganov '90, Bazhanov–Kashaev–Mangazeev–Stroganov '91].

Another L-operator

$L^{\mathrm{F}} =$ 

satisfies an RLL relation with Felder's R-matrix:



or

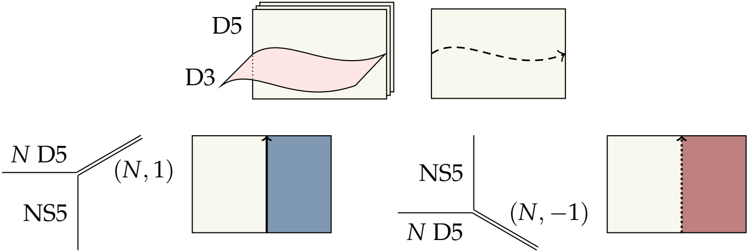
$$R^{\mathrm{F}} L_1^{\mathrm{F}} L_2^{\mathrm{F}} = L_2^{\mathrm{F}} L_1^{\mathrm{F}} R^{\mathrm{F}} .$$

L^{F} defines an ∞-dimensional representation of Felder's elliptic quantum group $E_{\tau,\gamma/2}(\mathfrak{sl}_N)$.

L^{B} gives a vertex-type elliptic algebra. Sklyanin algebra for $N = 2$.

Brane construction [Yamazaki '13, Maruyoshi–Y '16, Y '17]

Our model can be constructed from branes in String Theory:



We have

$$Z \text{ of the brane system} = Z \text{ of our model}\,.$$

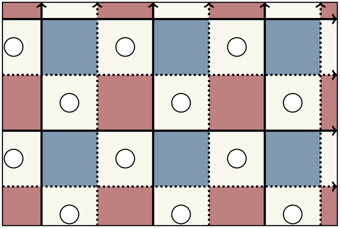
YBEs become brane movements.

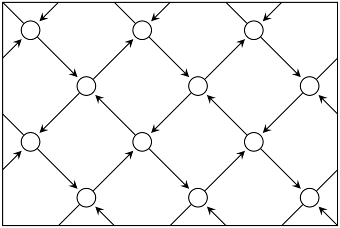
A 2d TQFT with "extra dimensions" underlies the correspondence [Costello '13, Y '15, '16].

Correspondence with 4d SUSY QFTs

The branes allow us to map our model to 4d SUSY QFT.

BS model $\leftrightarrow$ quiver gauge theory [Spiridonov '10, Yamazaki '13]:





Now ○ is an $\mathrm{SU}(N)$ gauge group, $\longrightarrow$ is a matter field.

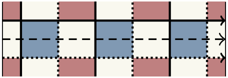
Z of this theory on $S^3 \times_{p,q} S^1 = Z_{\mathrm{BS}}$.

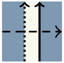
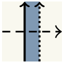
YBE = invariance of Z under Seiberg duality (quiver mutation).

Introduce an operator supported on a $T^2 \subset S^3 \times S^1$.

It acts on Z by a difference operator [Gadde–Gukov, Gaiotto–Rastelli–Razamat, Gaiotto–Razamat,…].

In the lattice model, it appears as a dashed lines [Maruyoshi–Y, Y '17]:



Recalling $L^{\mathrm{B}} =$  and $L^{\mathrm{F}} =$  , we can write it as

$$\mathrm{Tr}(L^{\mathrm{B}} \cdots L^{\mathrm{B}}) = \mathrm{Tr}(L^{\mathrm{F}} \cdots L^{\mathrm{F}}) \,.$$

This is a transfer matrix constructed from L^{B} or L^{F}.

Two choices for T^2 compatible with symmetries related by $p \leftrightarrow q$ lead to the elliptic modular double [Faddeev, Spiridonov].

By fusion (OPE of surface ops), we can construct a dashed line associated with any irrep R of $\mathfrak{sl}_N$:

$$R \dashrightarrow$$

$\mathrm{Tr}(\underbrace{L_R^{\mathrm{F}} \cdots L_R^{\mathrm{F}}}_{k})$ is a surface op for class-$\mathcal{S}_k$ theories [Gaiotto–Razamat].

For $R = \bigwedge^n V$ and $k = 1$, we get Ruijsenaars' ops [Hasegawa '95]. The proof uses the theta function identity

$$\sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \prod_{r=1}^{n} \left[\Phi\left(v+(r-1)\gamma, \lambda-\gamma \sum_{s=1}^{r-1} \omega_{i_s}\right)^{-1} \Phi\left(u+(r-1)\gamma, \lambda-\gamma \sum_{s=1}^{r-1} \omega_{i_{\sigma(s)}}\right)\right]_{i_{\sigma(r)}}^{i_r}$$
$$= \frac{\theta_1\big(v+(u-v)n/N\big)}{\theta_1(v)} \prod_{\substack{i\in I \\ k\notin I}} \frac{\theta_1\big(\lambda_{ki}+(u-v)/N\big)}{\theta_1(\lambda_{ki})} \; ; \quad I = \{i_1, \ldots, i_n\} \subset \{1, \ldots, N\}.$$

Reproduces the QFT result [Bullimore–Fluder–Hollands–Richmond '14].

The transfer matrices match QFT results [Maruyoshi–Y, Y '17] for

- $R = V, k = 1$ [Gaiotto–Rastelli–Razamat, Gadde–Gukov]
- $R = V, k > 1$ [Gaiotto–Razamat, Maruyoshi–Y, Ito–Yoshida]
- $R = \bigwedge^n V, n > 1, k = 1$ [Bullimore et al.]

Comparison in progress [Vaško–Y]:

- $R = S^n V, n > 1, k = 1$ [Gaiotto–Rastelli–Razamat, Gadde–Gukov]
- $R = S^n V, n > 1, k > 1$: partial results [Ito–Yoshida]

Need the symmetric analogue of Hasegawa's formula.

Other cases: no QFT results yet.

Conclusion

String Theory allows us to construct integrable lattice models and relate them to 4d SUSY QFTs.

Further directions:

- Change $S^3 \times S^1$ to $M_3 \times S^1$ to get new R-matrices: $M_3 = S^3/\mathbb{Z}_r$ [Yamazaki, Kels], $S^2 \times S^1$, $\Sigma \times S^1$, …
- Dimensional reduction [Y '15, Yamazaki–Wen, Gahramanov–Spiridonov, Gahramanov–Rosengren, Gahramanov–Kels]
- Relation to the works of Costello & Nekrasov–Shatashvili [Costello–Y, in progress]
- Zamolodchikov's tetrahedron equation [Y '15]
- Chiral Potts model and monopoles [Atiyah '91]
- Geometric Langlands and AGT correspondences
- Categorification of lattice models
- Little String Theory, AdS/CFT correspondence, …